Renormalization group for renormalization-group equations toward the universality classification of infinite-order phase transitions

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We derive a renormalization group to calculate the nontrivial critical exponent of the divergent correlation length, thereby providing a universality classification of essential singularities in infinite-order phase transitions. This method thus resolves the vanishing scaling matrix problem. The exponent is obtained from the maximal eigenvalue of a scaling matrix in this renormalization group, as in the case of ordinary second-order phase transitions. We exhibit several nontrivial universality classes in infinite-order transitions different from the well known Berezinskiĭ-Kosterlitz-Thouless transition. [S1063-651X(99)05010-2]

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I. INTRODUCTION

The Berezinskiĭ-Kosterlitz-Thouless (BKT) transition is well known as an infinite-order phase transition [1]. The correlation length ξ has an essential singularity at the critical coupling parameter g_c,

$$\xi \sim \exp(A|g-g_c|^{-\sigma}), \tag{1}$$

with the critical exponent $\sigma = 1/2$ or 1. In $c=1$ conformal field theory (CFT) [2], there are infinitely many models where infinite-order phase transitions can occur. Any one of them shows the same universality as the BKT transition.

One observes σ different from 1/2 or 1 in some $c>1$ CFTs [3–5]. Recently, a model of a quantum spin chain, whose long-distance behavior is described by the level-1 SU(N) Wess-Zumino-Witten (WZW) model, was studied by Itoi and Kato [3]. They pointed out that an infinite-order phase transition with a critical exponent $\sigma = N/(N+2)$ occurs by an SU(N) symmetry-breaking marginal operator. In the $N=3$ case, this corresponds to the gapless Haldane gap phase transition in a spin-1 isotropic antiferromagnet in one dimension. In a problem of dislocation-mediated melting, some curious numbers were observed by Young, Nelson, and Halperin [4]. They obtained $\sigma = 1/2$ for a model on a square lattice, $\sigma = 2/5$ for a simplified model on a triangular lattice and a nonalgebraic number $\sigma = 0.369\,63\ldots$ for a generalized model. In Ref. [5], Bulgadaev studied topological phase transitions in $c>1$ CFT with non-Abelian symmetry, where non-Abelian vortices play an important role. They belong to special classes of infinite-order phase transitions and several series of σ dependent on the symmetry of the system were found. Though there have been several studies on different types of models with infinite-order phase transitions including the BKT type, the universality classification by this critical exponent still remains a challenging problem.

In this paper, we study the universal nature of the critical exponent σ in infinite-order phase transitions. We show that the critical exponent σ is determined from the operator product coefficients of the marginal operators that cause the infinite-order phase transition. It is shown that a marginally irrelevant operator can also affect the value of the critical exponent σ.

The critical exponent of the correlation length is extracted from a long-distance asymptotic form of running coupling constants, whose leading term is determined by the motion of the coupling constants near a fixed point. In an ordinary finite-order phase transition, we linearize the renormalization-group equation (RGE) around the fixed point and can derive the exponent exactly. Namely, we can show that the inverse of the exponent is equal to the maximal eigenvalue of the scaling matrix defined by the derivative of the beta function at the fixed point. One does not have to solve the differential equation exactly in order to obtain the exact critical exponents in this case. In the infinite-order phase transition however, the scaling matrix vanishes, since the phase transition is driven only by marginal operators. So far, one has had to solve the differential equations explicitly to obtain the critical exponent, although RGEs with multiple variables are generally nonintegrable due to their nonlinearity except for some fortunate cases such as the BKT transition. This difficulty is one of the reasons why the universality classification of infinite-order phase transitions by the critical exponent σ in Eq. (1) has never been successfully done.

In order to resolve this problem, we apply another renormalization-group (RG) method developed in Refs. [6,7] to studying the long-distance asymptotic behavior of the solution of the original nonintegrable RGE. This RG method is starting to be recognized as a general tool for asymptotic analysis. Chen, Goldenfeld, and Oono [7] introduced the idea of RG to singular perturbation theory and gave a unified treatment. According to Bricmont, Kupiainen, and Lin [6], the RG transformation for a partial differential equation is

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defined as a semigroup transformation on a space of initial data, which is generated by a scaling transformation combined with time evolution. Koike, Hara, and Adachi used this general method in the study of the critical phenomenon in the Einstein equation of the gravitational collapse with formation of black holes [8]. Tasaki gave a pedagogical example of the RG transformation, where the equations of motion in Newtonian gravity were analyzed [9].

In Sec. II, we reuse the RG transformation of Ref. [9], which enables us to calculate the critical exponent σ in Eq. (1) without solving the nonlinear differential equation explicitly. Our RGE considers the straight flow line in the original RGE as a fixed point, where the derivative of the beta function in this RGE generally has nonzero value. In Sec. III, we show that the inverse of the maximal eigenvalue of the scaling matrix derived from this RGE gives the critical exponent σ. In Sec. IV, we also study asymptotic behavior of the running coupling constants in a massless phase and extend the well known formula for a logarithmic finite-size correction to the case of multiple running coupling constants. In Sec. V, we exhibit several nontrivial examples inspired by antiferromagnetic quantum spin chains. Finally, we give a summary and discussions in Sec. VI.

II. RGE FOR RGE

A. Formalism

Let us begin with the RGE for a given set of n marginal operators

$$\frac{d\boldsymbol{g}}{dt}=\boldsymbol{V}(\boldsymbol{g}), \tag{2}$$

where $\boldsymbol{g}=(g_1,\ldots,g_n)$ is a set of coupling parameters and $t=\log l$ with l being a length scale parameter. Since the operators are all marginal, the right-hand side is expanded as

$$V_k(\boldsymbol{g})=\sum_{ij} C_k^{ij} g_i g_j+O(g^3), \tag{3}$$

where C_k^{ij} is proportional to the operator product coefficients of the operators. First we neglect the higher-order terms $O(g^3)$, and later we discuss the irrelevance of the neglected terms.

In general, we find several critical surfaces where the RG flow is absorbed into the origin. A phase transition occurs if the initial coupling constants cross one of the critical surfaces. These critical surfaces divide the coupling parameter space into several regions which are phases. In the next section, we consider one massive phase surrounded by a set of critical surfaces, where there are several marginally relevant coupling parameters. In this region, we have a finite correlation length, which becomes larger as the coupling parameter approaches the critical surface.

We are going to study the long-distance asymptotic behavior of solutions for the RGE (2), which is generally nonintegrable. To this end, let us introduce the RG method explained in the Introduction. We define a renormalization-group transformation on an $n-1$-dimensional sphere that forms a set of initial values. We denote the solution $\boldsymbol{g}$ of Eq. (2) with the initial condition $\boldsymbol{a}_0=(a_{01},\ldots,a_{0n})$ as

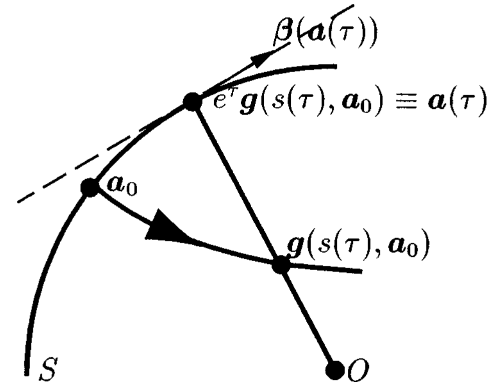

FIG. 1. Illustration of $\mathcal{R}_\tau$ and the β function defined in Eq. (10). For simplicity, we take $n=2$. The dashed line represents the tangent space at $\boldsymbol{a}(\tau)\in S$.

$$\boldsymbol{g}(t,\boldsymbol{a}_0), \tag{4}$$

namely, $\boldsymbol{g}(0,\boldsymbol{a}_0)=\boldsymbol{a}_0$. The function $e^\tau \boldsymbol{g}(e^\tau t,\boldsymbol{a}_0)$ is a solution of the RGE (2) as well, because of its scale invariance. Let S be the $n-1$-dimensional sphere whose center is at the origin with radius $|\boldsymbol{a}_0|\equiv a_0$. We define a new renormalization-group transformation $\mathcal{R}_\tau: S\rightarrow S$ as follows:

$$\mathcal{R}_\tau \boldsymbol{a}_0 \equiv e^\tau \boldsymbol{g}(s(\tau),\boldsymbol{a}_0)\equiv \boldsymbol{a}(\tau). \tag{5}$$

Note that $\mathcal{R}_\tau$ has a semigroup property:

$$\mathcal{R}_{\tau_1+\tau_2}=\mathcal{R}_{\tau_2}\circ\mathcal{R}_{\tau_1}. \tag{6}$$

The meaning of $\mathcal{R}_\tau$ is the following: first, choose τ. Then move $\boldsymbol{a}_0$ along the solution $\boldsymbol{g}(t,\boldsymbol{a}_0)$ during the time $s(\tau)$. Here $s(\tau)$ is determined by the condition $\boldsymbol{g}(s(\tau),\boldsymbol{a}_0)e^\tau\in S$. See Fig. 1.

Next let us derive the beta function for $\mathcal{R}_\tau$. Noting that $\boldsymbol{V}(\boldsymbol{g})$ is quadratic, we have

$$\frac{d\boldsymbol{a}}{d\tau}=\boldsymbol{a}+e^\tau \boldsymbol{V}(\boldsymbol{g}(s,\boldsymbol{a}_0))\frac{ds}{d\tau}$$

$$=\boldsymbol{a}+e^{-\tau}\boldsymbol{V}(\boldsymbol{a})\frac{ds}{d\tau}. \tag{7}$$

The length-preserving condition

$$\boldsymbol{a}\cdot\frac{d\boldsymbol{a}}{d\tau}=0 \tag{8}$$

leads to the following differential equation for $s(\tau)$:

$$\frac{ds}{d\tau}=-\frac{e^\tau a_0^2}{\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})}, \tag{9}$$

with the initial condition $s(0)=0$. Inserting Eq. (9) into Eq. (7), we obtain the beta function for $\mathcal{R}_\tau$:

$$\beta_i(\boldsymbol{a})\equiv\frac{da_i}{d\tau}=\frac{a_i\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})-V_i(\boldsymbol{a})a_0^2}{\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})}. \tag{10}$$

Note that $\boldsymbol{\beta}$ can be written as

$$\boldsymbol{\beta}(\boldsymbol{a}) = -\frac{a_0^2}{\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})} P(\boldsymbol{a})\boldsymbol{V}(\boldsymbol{a}), \tag{11}$$

where P is the $n\times n$ matrix that projects $\boldsymbol{V}(\boldsymbol{a}(\tau))$ onto the tangent space at $\boldsymbol{a}(\tau)\in S$:

$$P_{ij}(\boldsymbol{a}) \equiv \delta_{ij} - \frac{a_i a_j}{a_0^2}. \tag{12}$$

For later use, we derive a polar-coordinate representation of the new RGE. Employing polar coordinates, $\boldsymbol{a}\in S$ is expressed as

$$\boldsymbol{a} = \left(a_0 \prod_{\alpha=1}^{n-1} \sin\theta_\alpha, a_0\cos\theta_1 \prod_{\alpha=2}^{n-1}\sin\theta_\alpha, a_0\cos\theta_2\prod_{\alpha=3}^{n-1}\sin\theta_\alpha, \ldots, a_0\cos\theta_{n-1}\right). \tag{13}$$

Since $\{\partial\boldsymbol{a}/\partial\theta_\alpha\}_{1\leqslant\alpha\leqslant n-1}$ are orthogonal to each other, we can make the basis $\{\tilde{\boldsymbol{e}}_\alpha\}_\alpha$ orthonormal on the tangent space at $\boldsymbol{a}\in S$ by an appropriate rescaling:

$$\tilde{\boldsymbol{e}}_\alpha \equiv f_\alpha(\boldsymbol{a})^{-1}\frac{\partial\boldsymbol{a}}{\partial\theta_\alpha}, \quad f_\alpha(\boldsymbol{a}) \equiv \left|\frac{\partial\boldsymbol{a}}{\partial\theta_\alpha}\right|. \tag{14}$$

Then,

$$\tilde{\beta}_\alpha \equiv \boldsymbol{\beta}\cdot\tilde{\boldsymbol{e}}_\alpha = \frac{d\boldsymbol{a}}{d\tau}\cdot\tilde{\boldsymbol{e}}_\alpha = f_\alpha(\boldsymbol{a})\frac{d\theta_\alpha}{d\tau}, \tag{15}$$

which leads to the RGE in polar coordinates

$$\frac{d\theta_\alpha}{d\tau} = [f_\alpha(\boldsymbol{a})]^{-1}\tilde{\beta}_\alpha(\boldsymbol{a}). \tag{16}$$

Returning to the coordinate-free representation Eq. (11), let us find a fixed point of the new RGE (10). The nature of our RGE near the fixed point determines the universal behavior of the infinite-order phase transition, as in the ordinary RGE for a finite-order one. Near a fixed point, $\boldsymbol{a}(\tau)$ moves more slowly as its trajectory tends toward a critical surface. This implies that the time $\boldsymbol{a}(\tau)$ spent in a neighborhood of the fixed point is a singular function of the initial condition $\boldsymbol{a}_0$. This singularity can occur only at fixed points of the new RGE, which allows us to analyze its singular behavior by a linearization near the fixed points.

From Eq. (11), one finds that $\boldsymbol{a}$ is definitely a fixed point if $P(\boldsymbol{a})\boldsymbol{V}(\boldsymbol{a})=\boldsymbol{0}$ and $\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})\neq 0$. In this case, since $\boldsymbol{V}(k\boldsymbol{a})$ is parallel to $\boldsymbol{a}$ for all real numbers k, $\boldsymbol{a}$ is on a straight flow line of the original RGE (2). Straight flow lines are put into two classes. If an arbitrary point $\boldsymbol{a}$ on a straight flow line satisfies $\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})<0$, it is said to be an *incoming straight flow line* because $\boldsymbol{a}$ is carried toward the origin in time evolution. On the other hand, if $\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})>0$ for all $\boldsymbol{a}$ on a straight flow line, it is called an *outgoing straight flow line*. If a fixed point $\boldsymbol{a}$ of Eq. (10) is on an incoming straight flow line, $-\boldsymbol{a}$ is a fixed point on an outgoing straight flow line.

What happens if $P(\boldsymbol{a})\boldsymbol{V}(\boldsymbol{a})=\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})=0$? In this case, $\boldsymbol{V}(\boldsymbol{a})$ itself vanishes. It means that $\boldsymbol{a}$ is a fixed point of the original RGE (2). Moreover, since $\boldsymbol{V}$ is homogeneous, $k\boldsymbol{a}$ is also a fixed point for all $k\in\mathbf{R}$. Namely, the original RGE (2) has a fixed line in this case. If the original RGE (2) has this fixed line, a point on the fixed line has a nonvanishing scaling matrix even though the coupling constants are all marginal at the trivial fixed point $\boldsymbol{g}=\boldsymbol{0}$. Therefore, we can directly analyze the original RGE near a point on the fixed line and can show that the phase transition generally becomes of finite order in this case.

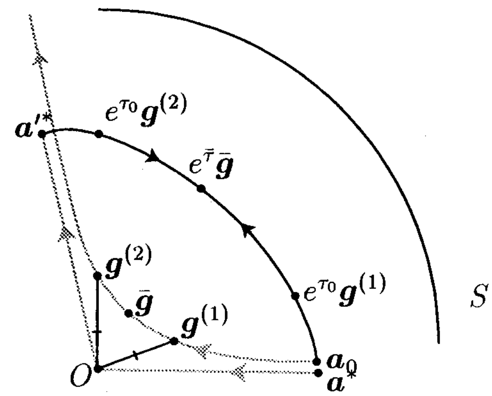

FIG. 2. An example in the case of a flow having a turning point. Here we take $n=3$. The gray lines represent solutions of the original RGE (2), while black ones on S represent our RGE (10). Here $\boldsymbol{a}^*(\boldsymbol{a}'^*)$ on the incoming (outgoing) straight flow line is a fixed point of our RGE.

Here, we offer a couple of remarks on the global nature of the new RG transformation $\mathcal{R}_\tau$ defined by Eq. (5). First, there could be a turning point $\bar{\boldsymbol{g}}$ where $\boldsymbol{V}(\bar{\boldsymbol{g}})\cdot\bar{\boldsymbol{g}}=0$ with $\boldsymbol{V}(\bar{\boldsymbol{g}})\neq\boldsymbol{0}$. Let

$$\ln\frac{a_0}{|\bar{\boldsymbol{g}}|} \equiv \bar{\tau}. \tag{17}$$

Although Eqs. (9) and (10) cannot be defined at $\tau=\bar{\tau}$, it is obvious in a geometric sense that $\boldsymbol{a}(\bar{\tau})$ and $s(\bar{\tau})$ are well defined. For example, in Fig. 2, $\boldsymbol{a}(\bar{\tau})=e^{\bar{\tau}}\bar{\boldsymbol{g}}$ and $s(\bar{\tau})$ are determined as the definite time $\boldsymbol{g}(t,\boldsymbol{a}_0)$ spent during the journey from $\boldsymbol{a}_0$ to $\bar{\boldsymbol{g}}$.

Second, if $\boldsymbol{g}(t,\boldsymbol{a}_0)$ has turning points, $\boldsymbol{a}(\tau)$ and $s(\tau)$ become multivalued with respect to τ. For example, in Fig. 2, $\boldsymbol{g}(t,\boldsymbol{a}_0)$ has a turning point at $\bar{\boldsymbol{g}}$. Suppose that $|\boldsymbol{g}^{(1)}|=|\boldsymbol{g}^{(2)}|=b$ and choose $\tau=\ln a/b\equiv\tau_0$. Then $\mathcal{R}_{\tau_0}\boldsymbol{a}_0$ has two images, $e^{\tau_0}\boldsymbol{g}^{(1)}$ and $e^{\tau_0}\boldsymbol{g}^{(2)}$. In this case we distinguish the images as $\boldsymbol{a}^{(1)}(\tau_0)$ and $\boldsymbol{a}^{(2)}(\tau_0)$. Similarly, $s(\tau_0)$ also has the same multiplicity, which is distinguished in a similar way. In Fig. 2, the image of $\mathcal{R}_\tau$ starting at $\boldsymbol{a}_0$ reaches a branch point $e^{\bar{\tau}}\bar{\boldsymbol{g}}$. We denote the solution from $\boldsymbol{a}_0$ to $e^{\bar{\tau}}\bar{\boldsymbol{g}}$ by $\boldsymbol{a}^{(1)}(\tau)$. The remaining part is called $\boldsymbol{a}^{(2)}(\tau)$. Both $\boldsymbol{a}^{(1)}(\tau)$ and $\boldsymbol{a}^{(2)}(\tau)$ are absorbed into the branch point $\bar{\boldsymbol{g}}e^{\bar{\tau}}$, which indicates the fact that $\bar{\boldsymbol{g}}$ gives a minimum distance from the origin. If a turning point corresponds to a maximum distance, the two solutions will escape from the branch point.

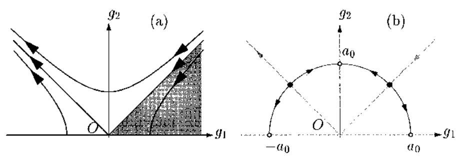

FIG. 3. (a) Flow generated by the original RGE (18) of the XY model. (b) Flow of our RGE (10) with Eq. (18). Fixed points are denoted by the black circles. Our RGE is not defined at the white circles, which correspond to a fixed line or turning points in Fig. 3(a).

B. Example — the two-dimensional classical XY model

Here we exhibit our RGE for the two-dimensional (2D) classical XY model as an illustrative example. The original RGE for the XY model is given by [10]

$$\begin{pmatrix} \dfrac{dg_1}{dt} \\ \dfrac{dg_2}{dt} \end{pmatrix} = \boldsymbol{V}(\boldsymbol{g}) \equiv \begin{pmatrix} -g_2^2 \\ -g_1 g_2 \end{pmatrix}, \tag{18}$$

with $g_2 \geqslant 0$. Let us first look at the phase structure from Eq. (18). See Fig. 3(a).

The RGE (18) has two straight flow lines $g_1 \pm g_2 = 0$ and one fixed line $g_2 = 0$. It is well known that each point on the fixed line corresponds to the theory of a 2D massless free boson that is parametrized by a compactification radius of the boson field [2]. The shaded region in Fig. 3(a) is a massless phase, since flow in this region is finally absorbed into a point on the $g_2 = 0$ fixed line. The incoming straight flow line, $g_1 - g_2 = 0$, with $g_2 \geqslant 0$, forms the phase boundary. As an initial coupling approaches the phase boundary from the massive phase, the correlation length ξ becomes divergent. (The correlation length also diverges when the initial coupling constants tend toward a fixed point on $g_2 = 0$ with $g_1 < 0$. However, the scaling matrix at this point does not vanish and the ordinary finite-order phase transition takes place. Since our interest is focused on an infinite-order phase transition, we do not consider that case here.)

Now we turn to the new RGE for the XY model, which is given by Eqs. (10) and (18) with the condition $a_1^2 + a_2^2 = a_0^2$ $(a_2 \geqslant 0)$. It is explicitly represented as

$$\begin{pmatrix} \dfrac{da_1}{d\tau} \\ \dfrac{da_2}{d\tau} \end{pmatrix} = \begin{pmatrix} \dfrac{1}{2a_1}(a_1^2 - a_2^2) \\ \dfrac{1}{2a_2}(a_2^2 - a_1^2) \end{pmatrix} \tag{19}$$

in Cartesian coordinates. Alternatively, using polar coordinates $\boldsymbol{a} = (a_0 \sin\theta,\ a_0 \cos\theta)$ $(-\pi/2 \leqslant \theta \leqslant \pi/2)$, the RGE becomes

$$\frac{d\theta}{d\tau} = -\cot 2\theta, \tag{20}$$

owing to Eq. (16).

Next we find fixed points of our new RGE. Solving

$$P(\boldsymbol{a})\boldsymbol{V}(\boldsymbol{a}) = \frac{1}{a_0^2} \begin{pmatrix} a_2^2(a_1^2 - a_2^2) \\ -a_1 a_2 (a_1^2 - a_2^2) \end{pmatrix} = \boldsymbol{0}, \tag{21}$$

we have $\boldsymbol{a} = (\pm a_0/\sqrt{2}, a_0/\sqrt{2}), (\pm a_0, 0)$. Evaluating $\boldsymbol{V}(\boldsymbol{a}) \cdot \boldsymbol{a}$ at those points, it turns out that $(a_0/\sqrt{2}, a_0/\sqrt{2})$ is on an incoming straight flow line while $(a_0/\sqrt{2}, -a_0/\sqrt{2})$ is on an outgoing straight flow line. The remaining points $(\pm a_0, 0)$ are found to be on a fixed line. Note that $\boldsymbol{V}(\boldsymbol{a}) \neq \boldsymbol{0}$ and $\boldsymbol{V}(\boldsymbol{a}) \cdot \boldsymbol{a} = 0$ when $\boldsymbol{a} = (0, a_0)$. This means that $(0, a_0)$ corresponds to turning points on trajectories generated by the original RGE. The flow of our RGE cannot be defined at the points $(0, a_0)$ and $(\pm a_0, 0)$. Since the example is a simple two-parameter system, we can understand qualitative aspects of the global flow in our RGE. See Fig. 3(b).

In the last part of the next section, we will continue the analysis of this model and derive σ from the β function in Eq. (19). Before doing that, we need to have a representation of the correlation length in terms of our RGE.

III. CRITICAL EXPONENT OF THE CORRELATION LENGTH IN A MASSIVE PHASE

In this section, we explain how to evaluate the critical exponent σ of the correlation length in a massive phase from the beta function (10).

We first define a correlation length $\xi(\boldsymbol{a}_0)$ by the following formula:

$$|\boldsymbol{g}(\ln \xi(\boldsymbol{a}_0), \boldsymbol{a}_0)| = 1. \tag{22}$$

Namely, $\ln \xi(\boldsymbol{a}_0)$ is the time $\boldsymbol{g}(t, \boldsymbol{a}_0)$ spent in the perturbative region. We note that $\xi(\boldsymbol{a}_0)$ defined above changes as

$$e^t \xi(\boldsymbol{g}(t, \boldsymbol{a}_0)) = \xi(\boldsymbol{a}_0) \tag{23}$$

under the original RG transformation, which should be satisfied by an intrinsic length scale of the model. The differential form of this equation,

$$\sum_i V_i(\boldsymbol{g}) \frac{\partial \xi(\boldsymbol{g})}{\partial g_i} + \xi(\boldsymbol{g}) = 0,$$

is obtained by Eq. (2), which is well known as an equation for an invariant length scale $\xi(\boldsymbol{g})$. Further, Eq. (22) is a natural generalization of the correlation length used in the 2D classical XY model.

We consider the case where the running coupling constants are in the perturbative region $|\boldsymbol{g}(t, \boldsymbol{a}_0)| < 1$. In this section, we study in particular the long-time asymptotics of a flow that approaches the origin once and then leaves for the nonperturbative region $|\boldsymbol{g}(t, \boldsymbol{a}_0)| \geqslant 1$, as the flow in the region $g_2 \geqslant |g_1|$ in the XY model. Generally, a quadratic differential equation such as Eq. (2) admits a flow qualitatively different from that investigated here. In Sec. VI, we discuss such an exceptional case.

Next, let us represent $\xi(\boldsymbol{a}_0)$ by the solution of our RGE (10). Define τ_{R} by $e^{-\tau_{\mathrm{R}}}|\boldsymbol{a}_0| = 1$. From the definition of $s(\tau)$, we obtain

$$\ln \xi(\boldsymbol{a}_0) = s(\tau_{\mathrm{R}}) = \int_0^{\tau_{\mathrm{R}}} d\tau \frac{ds}{d\tau}. \tag{24}$$

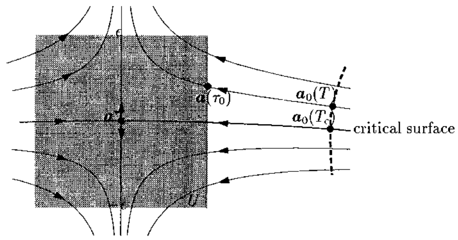

FIG. 4. Flow in our RGE (10) near the fixed point $\boldsymbol{a}^*$ on an incoming straight flow line. The dashed line represents the one-parameter family of the initial value $\boldsymbol{a}_0(T)$.

Using the differential equation (9) on the right-hand side, we obtain the integral representation for $\xi(\boldsymbol{a}_0)$. Since a flow treated in this section has a turning point as shown in Fig. 2, the correlation length is represented via

$$\ln \xi(\boldsymbol{a}_0) = -\int_0^{\bar{\tau}} d\tau \frac{e^{\tau} a^2}{\boldsymbol{a}^{(1)}(\tau)\cdot \boldsymbol{V}(\boldsymbol{a}^{(1)}(\tau))} - \int_{\bar{\tau}}^{\tau_R} d\tau \frac{e^{\tau} a^2}{\boldsymbol{a}^{(2)}(\tau)\cdot \boldsymbol{V}(\boldsymbol{a}^{(2)}(\tau))}. \quad (25)$$

Employing the integral representation, we argue that the leading term of ξ is given by

$$\ln \xi(\boldsymbol{a}_0) \simeq e^{\bar{\tau}} \quad (26)$$

if $\bar{\tau}$ in Eq. (17) is sufficiently large. Even though the integral near the turning point seems to diverge, it is only apparent as discussed in the preceding section [17]. The first term on the right-hand side of Eq. (25) diverges due to e^{τ} in the integrand when $\bar{\tau}$ goes to infinity. The second term contributes to the correlation length with the same order as the first term. Hence, we can evaluate $\xi(\boldsymbol{a}_0)$ by Eq. (26), which translates singular behavior of $\xi(\boldsymbol{a}_0)$ into that of $\bar{\tau}$.

Next, we evaluate the divergent $\bar{\tau}$ by using the polar-coordinate expression Eq. (16) of our RGE. It is obvious that $\bar{\tau}$ diverges if the initial coupling constant $\boldsymbol{a}_0$ is on an incoming straight flow line. It implies that $\bar{\tau}$ grows when $\boldsymbol{a}(\tau)$ passes near a fixed point $\boldsymbol{a}^*$ on the incoming straight flow line.

Suppose that $\boldsymbol{a}(\tau)$ goes through a neighborhood U of $\boldsymbol{a}^*$, as shown in Fig. 4.

The scaling matrix of the beta function (16) does not, in general, vanish at the fixed point and then the β function can be linearized in U. That is,

$$[f_\alpha(\boldsymbol{a})]^{-1}\tilde{\beta}_\alpha(\boldsymbol{a}) \simeq \sum_{\gamma=1}^{n-1} [f_\alpha(\boldsymbol{a}^*)]^{-1} \frac{\partial \tilde{\beta}_\alpha}{\partial \theta_\gamma}(\boldsymbol{a}^*)\,\delta\theta_\gamma \equiv \sum_{\gamma=1}^{n-1} A_{\alpha\gamma}(\boldsymbol{a}^*)\,\delta\theta_\gamma, \quad (27)$$

where $\delta\theta_\gamma \equiv \theta_\gamma - \theta_\gamma^*$ with $\{\theta_\gamma^*\}_\gamma$ representing the fixed point $\boldsymbol{a}^*$. The scaling matrix $A_{\alpha\gamma}(\boldsymbol{a}^*)$ describes the large $\bar{\tau}$ behavior because $\boldsymbol{a}(\tau)$ spends a long time in U. If the scaling matrix is diagonalized with eigenvalues b_α by a new coordinate $\{\theta'_\alpha\}_\alpha$, our RGE becomes

$$\frac{d}{d\tau}\delta\theta'_\alpha = b_\alpha \delta\theta'_\alpha. \quad (28)$$

The solution is

$$\delta\theta'_\alpha(\tau) = \delta\theta'_\alpha(\tau_0) e^{b_\alpha(\tau-\tau_0)}. \quad (29)$$

We take U as the $(n-1)$-dimensional cubic box, whose side is 2ϵ and whose center is $\boldsymbol{a}^*$. If the scaling matrix has the unique relevant mode θ'_1, $\boldsymbol{a}(\tau)$ spends time

$$\frac{1}{b_1}\ln\left|\frac{\epsilon}{\delta\theta'_1(\tau_0)}\right| \quad (30)$$

in U. Here we have supposed that $\boldsymbol{a}(\tau)$ reaches U at $\tau=\tau_0$. Now we vary the initial value $\boldsymbol{a}_0$ by one parameter T and assume that $\boldsymbol{a}_0(T)$ intersects a critical surface transversally at $T=T_c$. See Fig. 4.

As the initial value $\boldsymbol{a}_0(T)$ tends toward the critical surface, $|\delta\theta'_1(\tau_0)|$ gets small. It implies that $\delta\theta'_1(\tau_0)$ is expanded as

$$\delta\theta'_1(\tau_0) = \text{const}\times(T-T_c) + O[(T-T_c)^2]. \quad (31)$$

Since $\boldsymbol{a}(\tau)$ spends a finite time outside of U, the divergent behavior of $\xi(\boldsymbol{a}_0)$ is determined by Eqs. (30) and (31). Thus we get

$$\ln \xi(\boldsymbol{a}_0) \simeq e^{\bar{\tau}} \simeq |T-T_c|^{1/b_1}, \quad (32)$$

which means

$$\sigma = \frac{1}{b_1}. \quad (33)$$

This quantity does not depend strongly on the choice of $\boldsymbol{a}_0(T)$, so that σ is a universal quantity in this sense.

It is quite useful to find a relationship between the scaling matrix $A(\boldsymbol{a}^*)$ in Eq. (27) and the $n\times n$ matrix

$$B_{ij}(\boldsymbol{a}^*) \equiv \frac{\partial \beta_i}{\partial a_j}(\boldsymbol{a}^*) \quad (34)$$

for practical computing of the eigenvalues $\{b_\alpha\}_\alpha$. In the Appendix, we will show that

$$\Lambda(B) = \Lambda(A) \cup \{0\}, \quad (35)$$

where $\Lambda(M)$ is the set of eigenvalues of a matrix M. It should be noted that $B(\boldsymbol{a}^*) = B(-\boldsymbol{a}^*)$ since $\boldsymbol{\beta}(\boldsymbol{a})$ is an odd function. This means that the scaling matrix $A(\boldsymbol{a}^*)$ has the same eigenvalues as $A(-\boldsymbol{a}^*)$.

Now we deal with the 2D classical XY model again and show how to derive σ by our method. The original RGE is given by Eq. (18). We saw in the preceding section that $\boldsymbol{a}^* = (a_0/\sqrt{2}, a_0/\sqrt{2})$ is a fixed point on the incoming straight flow line. The matrix B defined by Eq. (34) is

$$B(\boldsymbol{a}^*)=\frac{\partial\beta_i}{\partial a_j}(\boldsymbol{a}^*)=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}\sim\begin{pmatrix}0 & 0\\ 0 & 2\end{pmatrix}, \tag{36}$$

i.e., the scaling matrix $A(\boldsymbol{a}^*)$ has the eigenvalue 2, according to Eq. (35). Alternatively, $A(\boldsymbol{a}^*)$ is directly computed in this case. From Eq. (20) we get

$$A(\boldsymbol{a}^*)=-\frac{1}{2}\frac{d}{d\theta}\bigg|_{\theta=\pi/4}\cot 2\theta=2, \tag{37}$$

as expected.

Following our result, Eq. (33), we get

$$\sigma=1/2, \tag{38}$$

which is well known as the BKT universality. Originally it is obtained integrating the nonlinear RGE (18) explicitly [10]. By contrast, according to our approach we can reach the same result in an algebraic way. As we will see in Sec. V, it can provide σ even in the case where an original RGE (2) is not integrable.

In the above example, the fixed point $\boldsymbol{a}^*$ has a unique relevant mode, so that we can apply the result (33). If there are multiple relevant modes in the scaling matrix, we can observe other relevant exponents $1/b_2, 1/b_3, \ldots >0$ in an appropriate fine tuning of the initial parameters.

Finally, we discuss the irrelevance of the higher-order terms in the original RGE (2). Here, we assume that we acquire no extra fixed points by taking into account higher-order terms. If we have higher-order terms, the RG transformed coupling with τ obeys a different equation because of their scale-breaking nature. The scaled coupling $\boldsymbol{g}'(t)=e^{\tau}\boldsymbol{g}(e^{\tau}t,\boldsymbol{a}_0)$ obeys

$$\frac{d\boldsymbol{g}'}{dt}=\boldsymbol{V}(\boldsymbol{g}')+O(e^{-\tau}\boldsymbol{g}'^3). \tag{39}$$

Note that the higher-order term becomes smaller and the RGE takes the scale-invariant form asymptotically. Therefore, higher-order terms are irrelevant in determining the critical exponent.

IV. LOGARITHMIC DEPENDENCE OF MULTIPLE MARGINALLY IRRELEVANT COUPLING CONSTANTS

So far, we have studied solutions of RGE (2) in a massive phase. Our method is also applicable to studying asymptotic behavior of a solution in a massless phase. In this section, we study the logarithmic dependence of the multiple running coupling constants in a massless phase.

It is important to clarify finite-size corrections in a system with marginally irrelevant operators. For example, a numerical simulation in a spin system can calculate energy levels only for small degrees of freedom. A theoretical formula for the finite-size correction is useful to extrapolate numerical data to those in the infinite system. If the system can be described in a critical theory with marginally irrelevant perturbations, physical quantities acquire logarithmic finite-size corrections. Here we are interested in a system with a finite volume L^D described by a theory with marginally irrelevant coupling constants $\boldsymbol{g}$, where D is the space dimension. Consider the situation where we have a critical theory with $\boldsymbol{g}=\boldsymbol{0}$ which describes the system with an infinite volume. In the system with a finite volume L^D, we can calculate physical quantities with a finite-size correction in the critical theory with a small perturbation of the coupling $\boldsymbol{g}$ obeying RGE (2). If we have an initial coupling $\boldsymbol{a}_0$ at a lattice spacing 1, the running coupling at the scale L becomes $\boldsymbol{g}(\ln L,\boldsymbol{a}_0)$, where $\boldsymbol{g}(\infty,\boldsymbol{a}_0)=\boldsymbol{0}$. In the case of a single marginally irrelevant coupling constant g, the original β function is given by

$$V(g)=Cg^2+O(g^3),$$

where C is a universal constant in the sense that it is independent of an initial value. The running coupling constant with an initial condition a_0 has the following solution:

$$\ln L=\int_{a_0}^{g(\ln L,a_0)}\frac{dg}{V(g)}\simeq\frac{1}{Cg(\ln L,a_0)}-\frac{1}{Ca_0}. \tag{40}$$

In this solution, we have a well known universal expression [11]

$$g(\ln L,a_0)=\frac{1}{C}\frac{1}{\ln L}+O\left(\frac{1}{(\ln L)^2},\frac{\ln\ln L}{(\ln L)^2}\right). \tag{41}$$

The leading term is independent of the initial coupling a_0, and therefore this formula is useful for fitting numerical or experimental data of the system with a finite size. For example, in one-dimensional quantum spin systems with marginally irrelevant perturbations, logarithmic finite-size corrections to the ground state energy

$$\Delta E_0=-\frac{\pi}{6L}\left[c+\frac{A}{(\ln L)^3}+O\left(\frac{\ln\ln L}{(\ln L)^4},\frac{1}{(\ln L)^4}\right)\right]$$

are calculated from this formula (41) [11,12], where c is the central charge and A is determined from C. Since c and A are universal constants, we can compare c and A to numerical (experimental) data and obtain a clue as to whether a field theory that derives Eq. (41) is or is not truly effective. Therefore, it is important to derive a formula corresponding to Eq. (41) where there are multiple marginally irrelevant couplings. In this section, we shall show that this universal nature holds in this case as well.

As we mentioned above, we examine the case where all the coupling constants are marginally irrelevant, so that flow of Eq. (2) is absorbed into the origin. In this case, there are no turning points on the flow, which implies that the transformation $\mathcal{R}_\tau\boldsymbol{a}_0$ defined in Eq. (5) is single-valued with respect to τ. Therefore, we can write down a formula similar to Eq. (25) as

$$\ln L=-\int_0^{\tau}d\tau'\frac{a_0^2e^{\tau'}}{\boldsymbol{a}(\tau')\cdot\boldsymbol{V}(\boldsymbol{a}(\tau'))}. \tag{42}$$

The running coupling constant $\boldsymbol{g}(\ln L,\boldsymbol{a}_0)$ is obtained by

$$\boldsymbol{g}(\ln L,\boldsymbol{a}_0)=e^{-\tau}\boldsymbol{a}(\tau). \tag{43}$$

In order to derive the logarithmic dependence of $\boldsymbol{g}(\ln L,\boldsymbol{a}_0)$, we first solve Eq. (42) for τ when L is sufficiently large. Then we apply the result to Eq. (43).

As we have seen in the preceding section, when we take L sufficiently large, the contribution from a neighborhood U of a fixed point $\boldsymbol{a}^*$ on an incoming straight flow line dominates in the integration of Eq. (42), which can be evaluated from the linearized new RGE in U. Suppose that $\boldsymbol{a}(\tau)$ enters U at $\tau=\tau_0$. Equation (42) becomes

$$\ln L \simeq -\int_{\tau_0}^{\tau} d\tau' \frac{a_0^2 e^{\tau'}}{\boldsymbol{a}(\tau')\cdot \boldsymbol{V}(\boldsymbol{a}(\tau'))} \tag{44}$$

for large L.

Writing

$$\boldsymbol{a}(\tau)=\boldsymbol{a}^*+\delta\boldsymbol{a}(\tau), \tag{45}$$

$\delta\boldsymbol{a}(\tau)$ in the polar-coordinate representation obeys the linearized RGE (28) in U. Its solution has the following asymptotic form for large τ:

$$\delta\boldsymbol{a}(\tau)\simeq \sum_{\alpha=1}^{n-1} \frac{\partial \boldsymbol{a}}{\partial \theta'_\alpha}\delta\theta'_\alpha(\tau)\simeq \frac{\partial \boldsymbol{a}}{\partial \theta'_1}\delta\theta'_1(\tau_0)e^{b_1(\tau-\tau_0)}, \tag{46}$$

where $b_1<0$ is the maximal eigenvalue of the scaling matrix $A(\boldsymbol{a}^*)$ defined in Eq. (27). We expand the integrand in Eq. (44) as

$$\frac{a_0^2 e^{\tau'}}{\boldsymbol{a}(\tau)\cdot \boldsymbol{V}(\boldsymbol{a}(\tau))} = \frac{a_0^2 e^{\tau'}}{\boldsymbol{a}^*\cdot \boldsymbol{V}(\boldsymbol{a}^*)}\left[1-\frac{(\delta\boldsymbol{a}\cdot\nabla|_{\boldsymbol{a}=\boldsymbol{a}^*})\boldsymbol{a}^*\cdot \boldsymbol{V}(\boldsymbol{a})}{\boldsymbol{a}^*\cdot \boldsymbol{V}(\boldsymbol{a}^*)}+O(\delta\boldsymbol{a}^2)\right], \tag{47}$$

and calculate the right-hand side of Eq. (44). First we compute the leading-order contribution. The leading integration is easily performed as follows:

$$\ln L \simeq -\int_{\tau_0}^{\tau} d\tau' \frac{a_0^2 e^{\tau'}}{\boldsymbol{a}^*\cdot \boldsymbol{V}(\boldsymbol{a}^*)} = -\frac{a_0^2}{\boldsymbol{a}^*\cdot \boldsymbol{V}(\boldsymbol{a}^*)}(e^{\tau}-e^{\tau_0})\simeq -\frac{a_0^2}{\boldsymbol{a}^*\cdot \boldsymbol{V}(\boldsymbol{a}^*)}e^{\tau}. \tag{48}$$

Since $\boldsymbol{a}\cdot \boldsymbol{V}(\boldsymbol{a})$, which is a cubic function of $\{a_k\}$, is negative at $\boldsymbol{a}^*$, we can write

$$\boldsymbol{a}^*\cdot \boldsymbol{V}(\boldsymbol{a}^*)=-Ca_0^3, \tag{49}$$

where C is a positive constant defined by

$$C\equiv -\boldsymbol{e}^*\cdot \boldsymbol{V}(\boldsymbol{e}^*) \tag{50}$$

with $\boldsymbol{e}^*\equiv \boldsymbol{a}^*/a_0$. From Eqs. (43), (48), and (49), we get, in the leading order,

$$\boldsymbol{g}(\ln L,\boldsymbol{a}_0)=e^{-\tau}\boldsymbol{a}^*\simeq \frac{1}{C\ln L}\boldsymbol{e}^*. \tag{51}$$

Since $\boldsymbol{e}^*$ and C are completely determined by the explicit form of $\boldsymbol{V}$, the result in the leading order is universal.

Next, let us go to the next-to-leading term. After evaluating the next-to-leading term in the integral (44) with the help of Eqs. (46) and (47), we represent e^{τ} in $1/\ln L$ expansion. The calculation is easily performed and finally we obtain

$$\boldsymbol{g}(\ln L,\boldsymbol{a}_0)=\begin{cases} \dfrac{1}{C\ln L}\boldsymbol{e}^*+\dfrac{B}{(\ln L)^{1-b_1}}\dfrac{\partial \boldsymbol{a}}{\partial\theta'_1} & (-1<b_1<0)\\[2ex] \dfrac{1}{C\ln L}\boldsymbol{e}^*+\dfrac{B'\ln\ln L}{(\ln L)^2}\dfrac{\partial \boldsymbol{a}}{\partial\theta'_1} & (b_1=-1)\\[2ex] \dfrac{1}{C\ln L}\boldsymbol{e}^*+\dfrac{B''}{(\ln L)^2}\boldsymbol{e}^* & (b_1<-1), \end{cases} \tag{52}$$

where the constants B, B', and B'' depend on the initial condition, in contrast to the leading term. The result implies that, if $-1\leqslant b_1<0$, we have to take into account the nonuniversal nature of the subleading correction, even though $O(g^3)$ corrections in the original renormalization-group equations give universal coefficients to this subleading term.

V. EXAMPLES

Here, we consider the level-1 $SU(N)$ Wess-Zumino-Witten (WZW) model in two dimensions as a critical theory [2]. This model has traceless chiral currents $J^{ab}(z)$ and $\overline{J}^{ab}(\overline{z})$ $(a,b=1,\ldots,N)$ satisfying the following operator product expansion (OPE):

$$J^{ab}(z)J^{cd}(0)\sim \frac{\delta_{ad}\delta_{bc}}{z^2}+\frac{1}{z}[\delta_{bc}J^{ad}(z)-\delta_{ad}J^{bc}(z)],$$

$$\overline{J}^{ab}(\overline{z})\overline{J}^{cd}(0)\sim \frac{\delta_{ad}\delta_{bc}}{\overline{z}^2}+\frac{1}{\overline{z}}[\delta_{bc}\overline{J}^{ad}(\overline{z})-\delta_{ad}\overline{J}^{bc}(\overline{z})],$$

$$J^{ab}(z)\overline{J}^{cd}(0)\sim 0, \tag{53}$$

where $\sim$ stands for equality up to regular terms. Using these currents, we can construct $(N^2-1)^2$ marginal operators $J^{ab}(z)\overline{J}^{cd}(\overline{z})$. In this section, we study models perturbed by some of the marginal operators which are inspired by a quantum spin chain [12,3].

A. Two-parameter system

First, we consider a simple two-parameter system that includes the BKT universality as the special case $N=2$. We define the $SU(N)$ symmetric marginal operator $\phi^1(z,\overline{z})$ and the symmetry breaking one $\phi^2(z,\overline{z})$,

$$\phi^1(z,\overline{z})=\sum_{a=1}^{N}\sum_{b=1}^{N} J^{ab}(z)\overline{J}^{ba}(\overline{z}),$$

$$\phi^2(z,\overline{z})=\sum_{a=1}^{N}\sum_{b=1}^{N} J^{ab}(z)\overline{J}^{ab}(\overline{z}), \tag{54}$$

which satisfy the following closed OPE:

$$\phi^1(z,\overline{z})\phi^1(0,0)\sim\frac{-2N}{|z|^2}\phi^1(0,0),$$

$$\phi^1(z,\overline{z})\phi^2(0,0)\sim\frac{-2}{|z|^2}[\phi^1(0,0)-\phi^2(0,0)],$$

$$\phi^2(z,\overline{z})\phi^2(0,0)\sim\frac{2N}{|z|^2}\phi^2(0,0) \quad (55)$$

by Eq. (53). The action integral $\mathcal{A}$ of the perturbed theory is

$$\mathcal{A}=\mathcal{A}_{\mathrm{WZW}}+\sum_{i=1}^{2} g_i\int\frac{d^2z}{2\pi}\phi^i(z,\overline{z}). \quad (56)$$

The OPE formula Eq. (55) yields the following two-parameter RGE:

$$\frac{dg_1}{dt}=V_1(\boldsymbol{g})=g_1(Ng_1+2g_2),$$

$$\frac{dg_2}{dt}=V_2(\boldsymbol{g})=-g_2(2g_1+Ng_2). \quad (57)$$

In the case of $N=2$, the RGE reduces to the same form as the RGE of the XY model with an appropriate linear transformation, which was extensively studied in Secs. II and III. Here we restrict ourselves to the case of $N\geqslant 3$.

The beta function in our RGE (10) for Eq. (57) is

$$\beta_1(\boldsymbol{a})=a_1-\frac{a_1(Na_1+2a_2)a_0^2}{a_1^2(Na_1+2a_2)-a_2^2(2a_1+Na_2)},$$

$$\beta_2(\boldsymbol{a})=a_2-\frac{-a_2(2a_1+Na_2)a_0^2}{a_1^2(Na_1+2a_2)-a_2^2(2a_1+Na_2)}. \quad (58)$$

Solving $\boldsymbol{\beta}(\boldsymbol{a})=\boldsymbol{0}$, we have the following six fixed points: $\pm(a_0,0)$, $\pm(0,a_0)$, and $\pm(a_0/\sqrt{2},-a_0/\sqrt{2})$. Evaluating $\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})$ at those points, we find that there are the three fixed points on incoming straight flow lines, $(-a_0,0)\equiv\mathbf{c}_1$, $(0,a_0)\equiv\mathbf{c}_2$, and $(-a_0/\sqrt{2},a_0/\sqrt{2})\equiv\mathbf{c}_3$. The matrix $B(\boldsymbol{a})$ in Eq. (34) at those points becomes

$$B(\mathbf{c}_1)=\begin{pmatrix}0 & 0\\ 0 & \dfrac{2+N}{N}\end{pmatrix},\quad B(\mathbf{c}_2)=\begin{pmatrix}\dfrac{2+N}{N} & 0\\ 0 & 0\end{pmatrix},$$

$$B(\mathbf{c}_3)=\frac{2+N}{4-2N}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}\sim\begin{pmatrix}0 & 0\\ 0 & \dfrac{2+N}{2-N}\end{pmatrix}, \quad (59)$$

which means $\mathbf{c}_1$ and $\mathbf{c}_2$ are unstable fixed points while $\mathbf{c}_3$ is stable for all $N\geqslant 3$.

Divergence of the correlation length is governed by the unstable fixed points and, according to the formula Eq. (33),

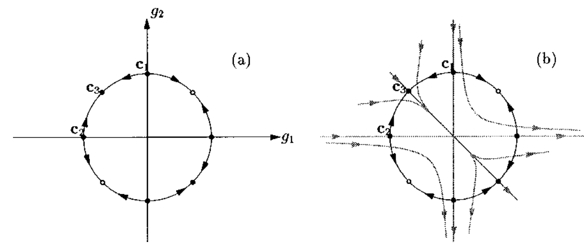

FIG. 5. (a) Flow in our RGE. The black circles represent fixed points while the white ones represent branch points corresponding to turning points on the flow in Eq. (57). (b) Illustration of the flow in Eq. (57) derived by our RGE, which is drawn in gray curves.

$$\sigma=\frac{N}{N+2}, \quad (60)$$

which is identical to that obtained by the explicit solution of the differential equation in Ref. [3]. Note that the result Eq. (60) is also valid in the case of $N=2$, although the scaling matrix cannot be defined at $\mathbf{c}_3$: $\mathbf{c}_3$ corresponds to a fixed line of the original RGE (57) if $N=2$.

Any theory of $c=1$ CFT with marginal perturbations has the critical exponent $\sigma=1/2$ or $\sigma=1$, as is well known [15]. This is because the level-1 $SU(2)$ WZW theory is the maximally symmetric theory in $c=1$ CFT and because it gives the most general theory with marginal perturbation in $c=1$ CFT [16]. The most general theory with marginal perturbations describes a quantum XYZ chain with spin 1/2. The infinite-order phase transition occurs at a line of the XXX chain. This corresponds to $N=2$ in our analysis. In the case of $c>1$ CFT with marginal perturbation, however, we show some new universality classes with nontrivial critical exponents σ Eq. (60) for $N>2$. For example, the transition in the case of $N=3$ describes the gapless Haldane gap phase transition with the exponent $\sigma=3/5$ from the SU(3) symmetric line $g_2=0$ in an isotropic spin-1 chain [3].

The result Eq. (59) is also useful when we figure out the qualitative picture of the flow in the original RGE (57). For this purpose, we need to know a branch point, which corresponds to a turning point on a flow in Eq. (57), by solving $\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})=0$. The solution is $\pm(a_0/\sqrt{2},a_0/\sqrt{2})$ for all $N\geqslant 3$. A flow in our RGE changes its direction at these points, as we depicted in Fig. 2. Combining the result Eq. (59) and the fact that the scaling matrix at $-\mathbf{c}_i$ $(i=1,2,3)$ has the same eigenvalues as at $\mathbf{c}_i$, we get the global flow of our RGE, as in Fig. 5(a). It derives qualitative features of the RG flow in Eq. (57), which are drawn in gray curves in Fig. 5(b). We notice that the region $g_1\langle 0,\ g_2\rangle$ is a massless phase, where solutions in the original RGE are all absorbed into the origin along the incoming straight flow line $g_1+g_2=0$ $(g_1<0)$. The incoming straight flow lines passing $\mathbf{c}_1$ or $\mathbf{c}_2$ form the phase boundary.

Next, we discuss logarithmic dependence of the running coupling constants in the massless phase. Let us introduce new variables $(X,Y)\equiv(g_1-g_2,-g_1-g_2)$. The original RGE has the incoming straight flow line $Y=0$ $(X<0)$ on which $\mathbf{c}_3$ is situated. According to our result Eq. (52), the running coupling constant $X(\ln L,\boldsymbol{a}_0)$ has the leading logarithmic dependence $1/\ln L$, whose coefficient is universal. In contrast, $Y(\ln L,\boldsymbol{a}_0)$ has the dependence $(\ln L)^{-1+b}$ with a nonuniversal coefficient, where $b\equiv(N+2)/(2-N)<-1$ for $N\geqslant 3$. Hence the $1/(\ln L)^2$ contribution that belongs to the next-to-leading term in $X(\ln L,\boldsymbol{a}_0)$ gives subleading contribution. This implies that, if we can determine g^3 terms in the original RGE (57), universal $\ln\ln L/(\ln L)^2$ dependences can be obtained in this example. We remark that the logarithmic dependence of $Y(\ln L,\boldsymbol{a}_0)$ is consistent with the result from the explicit solution [3].

B. Three-parameter system

Here, we consider a nontrivial three-parameter system, an SU(2)-invariant marginal perturbation of the level-1 SU(4) WZW model whose RGE becomes nonintegrable. This model may describe an $S=3/2$ quantum spin chain around the SU(4) symmetric Uimin-Lai-Sutherland model [13,14] with some SU(2) invariant perturbation. The SU(2) transformation is generated by

$$\mathrm{Tr}\int dz\, J(z)\boldsymbol{L}+\mathrm{Tr}\int d\bar{z}\,\overline{J}(\bar{z})\boldsymbol{L}, \tag{61}$$

where $\boldsymbol{L}=(L^1,L^2,L^3)$ is the spin matrix in the spin-3/2 representation. Marginal operators invariant under the SU(2) transformation are constructed as follows:

$$\phi^j(z,\bar{z})\equiv\mathrm{Tr}\sum_{m=-j}^{j}(-1)^m J(z)T_{j,m}\overline{J}(\bar{z})T_{j,-m},\quad j=0,1,2,3, \tag{62}$$

where $T_{j,m}$ satisfies $[\boldsymbol{L}^2,T_{j,m}]=j(j+1)T_{j,m}$ and $[L^3,T_{j,m}]=mT_{j,m}$. Using the tracelessness property of the currents, we get

$$\sum_{j=0}^{3}\phi^j(z,\bar{z})=0, \tag{63}$$

which indicates that there are three independent marginal operators in $\phi^0,\ldots,\phi^3$. Here we consider the perturbation

$$\sum_{j=0}^{2}g_i\int\frac{d^2z}{2\pi}\phi^j(z,\bar{z}). \tag{64}$$

Employing the OPE (53) and the normalization condition $\mathrm{Tr}\,{}^tT_{j,m}T_{j',m'}=\delta_{jj'}\delta_{mm'}$, we find the following operator product expansions:

$$\phi^0(z,\bar{z})\phi^0(0,0)\sim\frac{2}{|z|^2}\phi^0(0,0),$$

$$\phi^0(z,\bar{z})\phi^1(0,0)\sim\frac{1}{|z|^2}\left(-\frac{3}{5}\phi^0(0,0)-\frac{1}{2}\phi^1(0,0)-\frac{3}{10}\phi^2(0,0)\right),$$

$$\phi^0(z,\bar{z})\phi^2(0,0)\sim\frac{1}{|z|^2}\left(-\frac{1}{2}\phi^0(0,0)+\frac{1}{2}\phi^2(0,0)\right),$$

$$\phi^1(z,\bar{z})\phi^1(0,0)\sim\frac{1}{|z|^2}\left(-\frac{6}{25}\phi^0(0,0)+\frac{11}{5}\phi^1(0,0)+\frac{3}{5}\phi^2(0,0)\right),$$

$$\phi^1(z,\bar{z})\phi^2(0,0)\sim\frac{1}{|z|^2}\left(-\frac{9}{10}\phi^0(0,0)-\frac{1}{2}\phi^1(0,0)-\frac{6}{5}\phi^2(0,0)\right),$$

$$\phi^2(z,\bar{z})\phi^2(0,0)\sim\frac{1}{|z|^2}(3\phi^0(0,0)+3\phi^2(0,0)). \tag{65}$$

The RGE derived from the OPE is

$$\frac{dg_0}{dt}=V_0(\boldsymbol{g})=-g_0^2+\frac{3g_0g_1}{5}+3\frac{g_1^2}{25}+\frac{g_0g_2}{2}+\frac{9g_1g_2}{10}-\frac{3g_2^2}{2},$$

$$\frac{dg_1}{dt}=V_1(\boldsymbol{g})=\frac{g_0g_1}{2}-\frac{11g_1^2}{10}+\frac{g_1g_2}{2},$$

$$\frac{dg_2}{dt}=V_2(\boldsymbol{g})=\frac{3g_0g_1}{10}-\frac{3g_1^2}{10}-\frac{g_0g_2}{2}+\frac{6g_1g_2}{5}-\frac{3g_2^2}{2}. \tag{66}$$

We find that our RGE for Eq. (66) has the following seven fixed points on incoming straight flow lines:

$$\mathbf{c}_1=(a_0,0,0),\quad \mathbf{c}_2=\left(\frac{a_0}{\sqrt{2}},0,\frac{a_0}{\sqrt{2}}\right),\quad \mathbf{c}_3=\left(\frac{3a_0}{\sqrt{10}},0,\frac{a_0}{\sqrt{10}}\right),$$

$$\mathbf{c}_4=\left(3a_0\sqrt{\frac{2}{35}},a_0\sqrt{\frac{5}{14}},\frac{3a_0}{\sqrt{70}}\right),$$

$$\mathbf{c}_5=\left(\frac{-3a_0}{5\sqrt{26}},\frac{5a_0}{\sqrt{26}},\frac{2a_0}{5}\sqrt{\frac{2}{13}}\right),$$

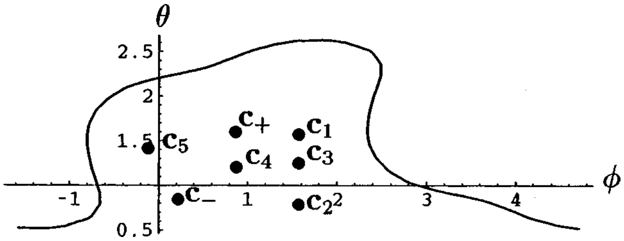

FIG. 6. Seven fixed points $\mathbf{c}_1, \ldots, \mathbf{c}_{\pm}$.

$$\mathbf{c}_{\pm}=\frac{a_0\sqrt{25105\mp 1747\sqrt{205}}}{\sqrt{5678}}\times\left(\frac{(\pm 239\sqrt{5}+85\sqrt{41})}{180},\frac{(\pm 13\sqrt{5}+5\sqrt{41})}{12},\mp\frac{1}{2\sqrt{5}}\right).$$

The corresponding eigenvalues of the scaling matrix at each fixed point are calculated as

$$\mathbf{c}_1;\ \left(\frac{3}{2},\frac{1}{2}\right),\quad \mathbf{c}_2;\ \left(\frac{3}{2},\frac{1}{2}\right),\quad \mathbf{c}_3;\ \left(\frac{5}{3},\frac{-1}{3}\right),$$

$$\mathbf{c}_4;\ (-5,-11),\quad \mathbf{c}_5;\ \left(\frac{55}{27},\frac{41}{27}\right),$$

$$\mathbf{c}_{\pm};\ \left(\frac{-4+\sqrt{631}}{8},\frac{-4-\sqrt{631}}{8}\right)=(2.63996417\ldots,-3.63996417\ldots).$$

Namely, they are classified into the twice-unstable fixed points $\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_5$, the once-unstable fixed points $\mathbf{c}_3,\mathbf{c}_{\pm}$, and the stable fixed point $\mathbf{c}_4$. The critical exponents determined by the once-unstable fixed points $\mathbf{c}_3$ and $\mathbf{c}_{\pm}$ are, respectively,

$$\sigma=\frac{3}{5},\quad \frac{8}{(-4+\sqrt{631})}, \tag{67}$$

which must be observed most likely in this model. In order to detect other exponents, fine tuning of the initial coupling parameters to the twice-unstable fixed points is necessary, as in the multicritical behavior of ordinary second-order phase transitions.

Logarithmic dependence of the running coupling constants is controlled by the stable fixed point $\boldsymbol{c}_4$. Since the largest eigenvalue at the fixed point is less than -1, the sub-leading contribution is $1/(\ln L)^2$ in this case.

It is helpful to study our RGE for Eq. (66) numerically in order to better understand our results. Figure 6 exhibits the seven fixed points $\mathbf{c}_1, \ldots, \mathbf{c}_{\pm}$ in polar coordinates,

$$\boldsymbol{a}=a_0(\sin\theta\sin\phi,\sin\theta\cos\phi,\cos\theta), \tag{68}$$

with $-\pi/2\leqslant\phi<3\pi/2$, $0\leqslant\theta<\pi$.

The curve in Fig. 6 represents the equation $\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})=0$, which corresponds to the set of branch points of the RG transformation $\mathcal{R}_\tau$. All fixed points belong to the region $\{\boldsymbol{a}|\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})<0\}$ because they are on incoming straight flow lines in the original RGE.

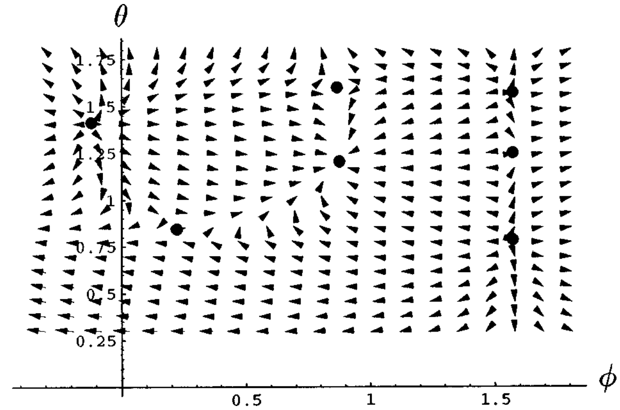

FIG. 7. Vector field defined by the polar-coordinate representation of our RGE for Eq. (66). The black dots represent the fixed points, which correspond to those in Fig. 6.

Figure 7 shows the vector field $(d\phi/d\tau, d\theta/d\tau)$ given by Eq. (16) near the fixed points. From this figure, we find that most points in this region go to the outside of the region or to the fixed point $\mathbf{c}_4$.

The destination of a trajectory toward the outside of this region is a point on the curve $\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})=0$. This implies that a flow of the original RGE (66) corresponding to this flow on S approaches the origin once and escapes from it. Namely, those points belong to a massive phase. On the other hand, a flow that terminates at the stable fixed point $\mathbf{c}_4$ corresponds to a flow of the original RGE absorbed into the origin along the incoming straight flow line. The set of those points moving toward the fixed point $\mathbf{c}_4$ is in a massless phase.

In general, we have two cases corresponding to massive and massless phases described above. However, there are exceptional points that lie on the solutions starting at one of the twice-unstable fixed points $\mathbf{c}_1,\mathbf{c}_2,\mathbf{c}_5$ and arriving at one of the once-unstable fixed points $\mathbf{c}_3,\mathbf{c}_{\pm}$. These exceptional solutions correspond to critical surfaces that give the phase boundary determined in the original RGE. More precisely, the set of points on the exceptional trajectories is the intersection between a phase boundary determined by the original RGE and the sphere S. It should be noted that the phase boundary in the coupling space $\{(g_1,g_2,g_3)\}$ forms a conical surface because $\boldsymbol{V}(\boldsymbol{g})$ is homogeneous.

Figure 8 depicts a flow numerically solved near the criti-

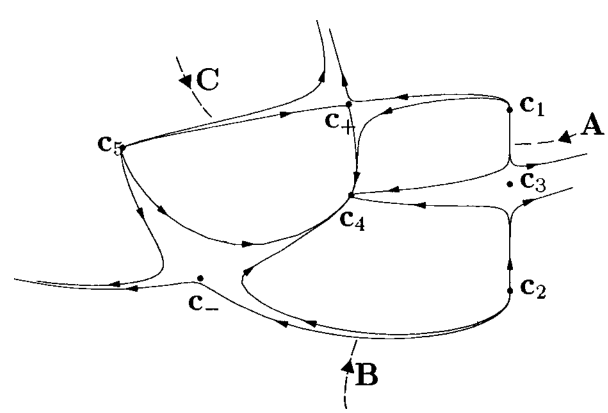

FIG. 8. Solutions numerically computed near the exceptional ones.

cal surfaces. Using the numerical results, Figs. 7 and 8, let us consider divergent behavior of the correlation length. Suppose that an initial value $\boldsymbol{a}_0$ changes toward the phase boundary between $\mathbf{c}_1$ and $\mathbf{c}_2$ such as the dashed line **A**. As $\boldsymbol{a}_0$ tends to the phase boundary, the flow starting at $\boldsymbol{a}_0$ passes near the fixed point $\mathbf{c}_3$ and spends a long time there. The result Eq. (67) indicates that the critical exponent σ detected in this case is 3/5. Similarly, if $\boldsymbol{a}_0$ varies along lines such as **B** or **C**, one observes $\sigma=8/(-4+\sqrt{631})$, since the solution starting at $\boldsymbol{a}_0$ goes through a neighborhood of the fixed points $\mathbf{c}_+$ or $\mathbf{c}_-$.

Finally, we comment on the logarithmic dependence of the coupling constants in this example. Figure 7 implies that a general massless flow is controlled by the twice-stable fixed point $\mathbf{c}_4$. The universal features considered in Sec. IV are valid if the initial value $\boldsymbol{a}_0$ is sufficiently far from the phase boundary. However, as $\boldsymbol{a}_0$ approaches the phase boundary from a massless side, the solution is gradually affected by the once-unstable fixed points. More profound investigation will be needed in this case.

VI. SUMMARY AND DISCUSSION

In this paper, we first showed an algebraic way of finding the critical exponent σ in Eq.(1), which was heretofore computed by integrating RGE explicitly. The procedure is summarized as follows.

(i) Derive the RGE defined in Eq. (10) from the original RGE (2).

(ii) Find straight flow lines in the original RGE, which correspond to fixed points of our RGE.

(iii) Compute the scaling matrix at a fixed point on an incoming straight flow line and diagonalize it.

(iv) If the scaling matrix has the unique relevant mode, the correlation length indicates singular behavior by one-parameter fine tuning and the exponent σ is equal to the inverse of the relevant eigenvalue. If the scaling matrix has multiple relevant modes, we can observe multicritical behavior.

Second, we derived the logarithmic dependence of running coupling constants in a massless phase where all the coupling constants are marginally irrelevant. It was found that the coefficient of the leading logarithmic term is universal in the sense that it does not depend on an initial value of the running coupling constants, which is the same result as in the case of a single marginally irrelevant coupling constant. However, coefficients of subleading terms are nonuniversal. They could disturb the universal nature of subleading terms that come from higher-loop corrections to the original beta function.

We obtain both results by applying the RG transformation (5) to the original RGE (2), which was inspired by the recent developments of RG transformations to nonlinear differential equations [6–8].

It should be noted that our study is focused, when we derive the first result, on a flow that goes once toward the origin and then leaves for a nonperturbative region. In general, the quadratic differential equation (2) with Eq. (3) could have a flow qualitatively different from those we considered. For example, we numerically find that the equation

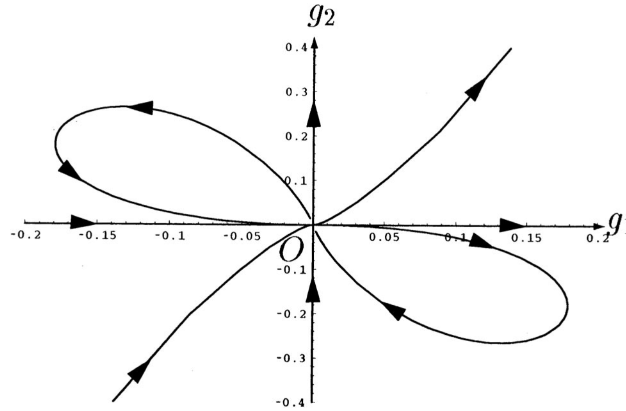

FIG. 9. Flow in Eq. (69) with $q=2$.

$$\begin{pmatrix} \dfrac{dg_1}{dt} \\ \dfrac{dg_2}{dt} \end{pmatrix} = \boldsymbol{V}(\boldsymbol{g}) \equiv \begin{pmatrix} g_1(g_1+g_2) \\ g_2(qg_1+g_2) \end{pmatrix} \tag{69}$$

has a flow such as in Fig. 9 if $q>1$.

Namely, the equation has solutions that first escape from the origin and then turn back to it. See Fig. 10. In this case we cannot directly apply the result Eq. (33). Even in that case, as we show in the following, the beta function in Eq. (10) is helpful for understanding a qualitative picture of these solutions. In fact, we first notice that there are only four straight flow lines on $g_1=0$ or on $g_2=0$. Next let us compute the scaling matrix in Eq. (27) at the fixed points $(\pm a_0,0)$ and $(0,\pm a_0)$ of our RGE (10). We find that it has eigenvalues $1-q$ at $(\pm a_0,0)$ and 0 at $(0,\pm a_0)$. If $q>1$, solutions for the RGE near $(\pm a_0,0)$ is absorbed into the fixed points as depicted in Fig. 10. The integral curve does not change the direction at the other fixed points $(0,\pm a_0)$ because the eigenvalue of the scaling matrix vanishes at those points. Therefore, by continuity, we conclude that there must be at least two branch points on S. Moreover, the solution must escape from the branch points. As we pointed out in the last part of II A, it means that Eq. (69) has a flow that first leaves from the origin and turns back to it, as depicted in Fig. 9.

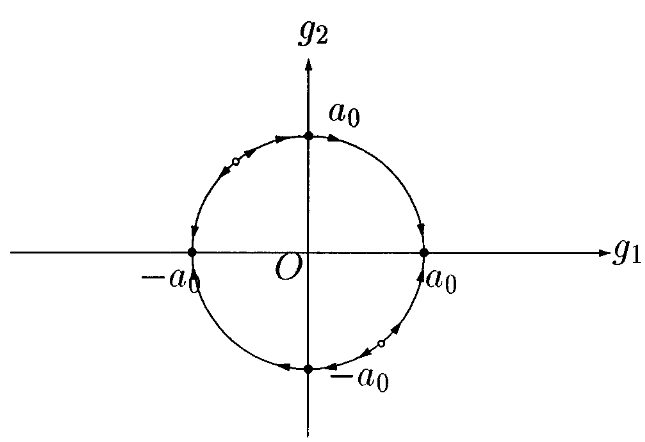

FIG. 10. Flow in our RGE for Eq. (69) with $q=2$. The white circles correspond to turning points.

The study of the universality classification of infinite-order phase transitions is now in progress. In two dimensions, it should contribute to the analysis of the $c>1$ CFT. In higher than two dimensions, a nontrivial exponent in an infinite-order phase transition might be observed experimentally in some phenomena.

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APPENDIX: RELATIONSHIP BETWEEN THE EIGENVALUES OF $A(\boldsymbol{a}^*)$ AND $B(\boldsymbol{a}^*)$

In this Appendix we prove Eq. (35). The claim is that a set of the eigenvalues of $B(\boldsymbol{a}^*)$ in Cartesian coordinates is equal to that of $A(\boldsymbol{a}^*)$ in polar coordinates plus extra zero eigenvalue.

To show that, we add $\boldsymbol{a}/a\equiv\tilde{\boldsymbol{e}}_n$ to the basis $\{\tilde{\boldsymbol{e}}_\alpha\}_{1\leqslant\alpha\leqslant n-1}$ of the tangent space at $\boldsymbol{a}(\tau)\in S$. Then the set $\{\tilde{\boldsymbol{e}}_i\}_{1\leqslant i\leqslant n}$ becomes an orthonormal basis of the n-dimensional space of coupling constants. Consider the $n\times n$ matrix

$$T_{ij}(\boldsymbol{a})\equiv f_j(\boldsymbol{a})(\boldsymbol{e}_i,\tilde{\boldsymbol{e}}_j), \tag{A1}$$

where $\{\boldsymbol{e}_i\}_{1\leqslant i\leqslant n}$ is the orthonormal basis that defines the Cartesian coordinates $(g_1,\ldots,g_n)$ and the bracket $(\boldsymbol{x},\boldsymbol{y})$ means the inner product. The function $f_j(\boldsymbol{a})$ is given in Eq. (14) for $1\leqslant j\leqslant n-1$. In addition,

$$f_n(\boldsymbol{a})\equiv\left|\frac{\partial\boldsymbol{a}}{\partial a}\right|=1. \tag{A2}$$

Since $(\boldsymbol{e}_i,\tilde{\boldsymbol{e}}_j)$ forms an orthogonal matrix, we immediately have the inverse of T as

$$T^{-1}_{ik}(\boldsymbol{a})=f_i(\boldsymbol{a})^{-1}(\tilde{\boldsymbol{e}}_i,\boldsymbol{e}_k). \tag{A3}$$

Now we examine the form of the $n\times n$ matrix $T^{-1}BT$. As the first step, let us compute $T^{-1}B$:

$$\begin{aligned}\sum_{k=1}^{n} T^{-1}_{ik}(\boldsymbol{a}^*)B_{kl}(\boldsymbol{a}^*)&=\sum_{k=1}^{n} f_i(\boldsymbol{a}^*)^{-1}(\tilde{\boldsymbol{e}}_i,\boldsymbol{e}_k)\frac{\partial\beta_k}{\partial a_l}(\boldsymbol{a}^*)\\&=f_i(\boldsymbol{a}^*)^{-1}\frac{\partial}{\partial a_l}(\tilde{\boldsymbol{e}}_l,\boldsymbol{\beta})\\&=f_i(\boldsymbol{a}^*)^{-1}\frac{\partial\tilde{\beta}_i}{\partial a_l}(\boldsymbol{a}^*).\end{aligned} \tag{A4}$$

Here we have used $\boldsymbol{\beta}(\boldsymbol{a}^*)=\boldsymbol{0}$ in the second equality. Next, according to Eq. (14), we find that T_{ij} can be written as

$$T_{lj}=\left(\boldsymbol{e}_l,\frac{\partial\boldsymbol{a}}{\partial\theta_j}\right)=\frac{\partial a_l}{\partial\theta_j}\qquad(1\leqslant j\leqslant n-1). \tag{A5}$$

Using Eqs. (A4) and (A5), we get

$$\begin{aligned}&\sum_{1\leqslant k,l\leqslant n} T^{-1}_{ik}(\boldsymbol{a}^*)B_{kl}(\boldsymbol{a}^*)T_{lj}(\boldsymbol{a}^*)\\&\quad=\sum_{1\leqslant k,l\leqslant n} f_j(\boldsymbol{a}^*)^{-1}\frac{\partial\tilde{\beta}_i}{\partial a_l}(\boldsymbol{a}^*)\frac{\partial a_l}{\partial\theta_j}(\boldsymbol{a}^*)\\&\quad=f_j(\boldsymbol{a}^*)^{-1}\frac{\partial\tilde{\beta}_i}{\partial\theta_j}(\boldsymbol{a}^*)\end{aligned} \tag{A6}$$

for $1\leqslant i\leqslant n$ and $1\leqslant j\leqslant n-1$.

The result indicates that $T^{-1}BT$ has the following form:

$$(T^{-1}BT)(\boldsymbol{a}^*)=\begin{pmatrix}A(\boldsymbol{a}^*) & *\\ 0\cdots0 & 0\end{pmatrix}, \tag{A7}$$

which proves Eq. (35). Note that the last row vanishes because $\tilde{\beta}_n(\boldsymbol{a})=[\boldsymbol{\beta}(\boldsymbol{a}),\tilde{\boldsymbol{e}}_n]=0$ for all $\boldsymbol{a}$.

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[17] In fact, there exists a component of $\boldsymbol{V}$ that does not vanish at the turning point, say, V_1. We parametrize the flow of our RGE by a_1 instead of τ near the turning point and change the integration variable in Eq. (25). The measure changes as

$$\frac{d\tau}{\boldsymbol{a}(\tau)\cdot\boldsymbol{V}(\boldsymbol{a}(\tau))}=\frac{da_1}{a_1\boldsymbol{a}\cdot\boldsymbol{V}(\boldsymbol{a})-a_0^2V_1(\boldsymbol{a})}$$

from Eq. (10). The denominator of the right-hand side does not vanish near the turning point. Therefore, the denominators in the integrand in Eq. (25) does not contribute to the divergence of $\xi(\boldsymbol{a}_0)$ and can be replaced by a certain constant when we evaluate the leading divergence of the integration in Eq. (25).